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Non-Noether symmetries in integrable models

George Chavchanidze

Department of Theoretical Physics, A Razmadze Institute of Mathematics, 1 Aleksidze Street, Tbilisi 0193, Georgia

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Abstract
In the present paper the non-Noether symmetries of the Toda model, nonlinear Schrödinger equation and Korteweg–de Vries equations (KdV and mKdV) are discussed. It appears that these symmetries yield the complete sets of conservation laws in involution and lead to the bi-Hamiltonian realizations of the above-mentioned models.

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Because of their exceptional properties the non-Noether symmetries could be effectively used in analysis of Hamiltonian dynamical systems. From the geometric point of view these symmetries are important because of their tight relationship with geometric structures on phase space such as bi-Hamiltonian structures, Frölicher–Nijenhuis operators, Lax pairs and bicomplexes [1]. The correspondence between non-Noether symmetries and conservation laws is also interesting and in regular Hamiltonian systems on $2n$-dimensional Poisson manifold up to n integrals of motion could be associated with each generator of non-Noether symmetry [1, 3]. As a result non-Noether symmetries could be especially useful in analysis of Hamiltonian systems with many degrees of freedom, as well as infinite-dimensional Hamiltonian systems, where a large (and even infinite) number of conservation laws could be constructed from the single generator of such a symmetry. Under certain conditions satisfied by the symmetry generator these conservation laws appear to be involutive and ensure integrability of the dynamical system.

The n-particle non-periodic Toda model is one of integrable models that possesses such a nontrivial symmetry. In this model non-Noether symmetry (which is a one-parameter group of noncannonical transformations) yields conservation laws that appear to be functionally independent, involutive and ensure the integrability of this dynamical system. The well-known bi-Hamiltonian realization of the Toda model is also related to this symmetry.

The nonlinear Schrödinger equation is another important example where symmetry (again a one-parameter group) leads to the infinite sequence of conservation laws in involution. The KdV and mKdV equations also possess non-Noether symmetries which are quite nontrivial (but the symmetry group is still one-parameter) and in each model the infinite set of conservation laws is associated with the single generator of the symmetry.

Before we consider these models in detail we briefly recall some basic facts concerning symmetries of Hamiltonian systems. Since throughout the paper continuous one-parameter groups of symmetries play a central role, let us recall that each vector field E on the phase space M of the Hamiltonian dynamical system defines a continuous one-parameter group of transformations (flow)

$$g_a = \mathrm{e}^{aL_E} \tag{1}$$

where L_E denotes the Lie derivative along the vector field E. The action of this group on observables (smooth functions on M) is given by the expansion

$$g_a(f) = \mathrm{e}^{aL_E}(f) = f + aL_E f + \tfrac{1}{2}a^2 L_E^2 f + \cdots. \tag{2}$$

Further it will be assumed that M is a $2n$-dimensional symplectic manifold and the group of transformations g_a will be called a symmetry of the Hamiltonian system if it preserves the manifold of solutions of Hamilton's equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{h, f\} \tag{3}$$

(here {,} denotes the Poisson bracket defined in a standard manner by the Poisson bivector field $\{f, g\} = W(\mathrm{d}f \wedge \mathrm{d}g)$ and h is a smooth function on M called the Hamiltonian) or in other words if for each f satisfying Hamilton's equation $g_a(f)$ also satisfies it. This happens when g_a commutes with the time evolution operator

$$\frac{\mathrm{d}}{\mathrm{d}t} g_a(f) = g_a\left(\frac{\mathrm{d}}{\mathrm{d}t} f\right). \tag{4}$$

If in addition the generator E of the group g_a does not preserve Poisson bracket structure $[E, W] \neq 0$ then the g_a is called non-Noether symmetry. Note that bracket [,] known as the Schouten bracket or supercommutator is actually a graded extension of the ordinary commutator of vector fields to the case of multivector fields, and could be defined by linearity and derivation property

$$\begin{aligned}&[C_1 \wedge C_2 \wedge \ldots \wedge C_n, S_1 \wedge S_2 \wedge \ldots \wedge S_n]\\ &= (-1)^{p+q}[C_p, S_q] \wedge C_1 \wedge C_2 \wedge \ldots \wedge \hat{C}_p \wedge \ldots \wedge C_n \wedge S_1 \wedge S_2 \wedge \ldots \wedge \hat{S}_q \wedge \ldots \wedge S_n\end{aligned} \tag{5}$$

where an overhat denotes the omission of the corresponding vector field.

Let us briefly recall some basic features of non-Noether symmetries. First of all, if E generates non-Noether symmetry then the n functions

$$Y_k = i_{W^k}(L_E\omega)^k \qquad k = 1, 2, \ldots, n \tag{6}$$

(where ω is the symplectic form obtained by inverting Poisson bivector W and i denotes the contraction) are integrals of motion (see [1, 3]). Note that in the finite-dimensional case, conservation laws (6) can be calculated using the formula

$$Y_k = \frac{n!}{(n-k)!k!} \frac{\hat{W}^k \wedge W^{n-k}}{W^n} \qquad k = 1, 2, \ldots, n \tag{7}$$

where $\hat{W} = [E, W]$. The ratio of multivectors is defined correctly since the space of maximal degree multivectors is one dimensional. The advantage of this expression is that there is no need to invert W, but unfortunately it can be used only in the finite-dimensional case (otherwise multivectors such as W^n do not exist). It is also known that if additionally the symmetry generator E satisfies the Yang–Baxter equation

$$[[E[E, W]]W] = 0 \tag{8}$$

conservation laws Y_k appear to be in involution $\{Y_k, Y_m\} = 0$ while the bivector fields W and $\hat{W} = [E, W]$ (or in terms of 2-forms ω and $L_E\omega$) form a bi-Hamiltonian system (see [1]). Due to these features non-Noether symmetries could be effectively used in construction of conservation laws and bi-Hamiltonian structures.

Now let us focus on non-Noether symmetry of the Toda model—a $2n$-dimensional Hamiltonian system that describes the motion of n particles on the line governed by the exponential interaction. The equations of motion of the non-periodic n-particle Toda model are

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = p_i \qquad \frac{\mathrm{d}}{\mathrm{d}t} p_i = \epsilon(i-1)\,\mathrm{e}^{q_{i-1}-q_i} - \epsilon(n-i)\,\mathrm{e}^{q_i-q_{i+1}} \tag{9}$$

($\epsilon(k) = -\epsilon(-k) = 1$ for any natural k and $\epsilon(0) = 0$) and could be rewritten in Hamiltonian form (3) with canonical Poisson bracket defined by

$$W = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \tag{10}$$

corresponding symplectic form

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i \tag{11}$$

and Hamiltonian equal to

$$h = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i-q_{i+1}}. \tag{12}$$

The group of transformations g_a generated by the vector field E will be a symmetry of the Toda chain if for each p_i, q_i satisfying Toda equations (9) $g_a(p_i)$, $g_a(q_i)$ also satisfy it. Substituting infinitesimal transformations

$$g_a(p_i) = p_i + aE(p_i) + O(a^2) \qquad g_a(p_i) = q_i + aE(q_i) + O(a^2) \tag{13}$$

into (9) and grouping first-order terms gives rise to the conditions

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} E(q_i) = E(p_i) \\ &\frac{\mathrm{d}}{\mathrm{d}t} E(p_i) = \epsilon(i-1)\,\mathrm{e}^{q_{i-1}-q_i}(E(q_{i-1}) - E(q_i)) - \epsilon(n-i)\,\mathrm{e}^{q_i-q_{i+1}}(E(q_i) - E(q_{i+1})). \end{aligned} \tag{14}$$

One can verify that the vector field defined by

$$\begin{aligned} E(p_i) &= \frac{1}{2}p_i^2 + \epsilon(i-1)(n-i+2)\,\mathrm{e}^{q_{i-1}-q_i} - \epsilon(n-i)(n-i)\,\mathrm{e}^{q_i-q_{i+1}} \\ &\quad + \frac{t}{2}(\epsilon(i-1)(p_{i-1}+p_i)\,\mathrm{e}^{q_{i-1}-q_i} - \epsilon(n-i)(p_i+p_{i+1})\,\mathrm{e}^{q_i-q_{i+1}}) \\ E(q_i) &= (n-i+1)p_i - \frac{1}{2}\sum_{k=1}^{i-1} p_k + \frac{1}{2}\sum_{k=i+1}^{n} p_k \\ &\quad + \frac{t}{2}\left(p_i^2 + \epsilon(i-1)\,\mathrm{e}^{q_{i-1}-q_i} + \epsilon(n-i)\,\mathrm{e}^{q_i-q_{i+1}}\right) \end{aligned} \tag{15}$$

satisfies (14) and generates symmetry of the Toda chain. It appears that this symmetry is non-Noether since it does not preserve Poisson bracket structure $[E, W] \neq 0$ and additionally one can check that Yang–Baxter equation $[[E[E, W]]W] = 0$ is satisfied. This symmetry could play an important role in the analysis of the Toda model. First let us note that calculating $L_E\omega$ leads to the following 2-form,

$$L_E\omega = \sum_{i=1}^{n} p_i \,\mathrm{d}p_i \wedge \mathrm{d}q_i + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \,\mathrm{d}q_i \wedge \mathrm{d}q_{i+1} + \sum_{i<j} \mathrm{d}p_i \wedge \mathrm{d}p_j \tag{16}$$

and together ω and $L_E\omega$ give rise to bi-Hamiltonian structure of the Toda model (compare with [2]). By taking the Lie derivative of the Poisson bivector field W along symmetry one reproduces the second bivector field

$$\hat{W} = [E, W] = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{i+1}} + \sum_{i<j} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_j} \tag{17}$$

involved in the bi-Hamiltonian realization of the Toda chain. The conservation laws (6) associated with the symmetry reproduce the well-known set of conservation laws of the Toda chain,

$$\begin{aligned}
I_1 &= Y_1 = \sum_{i=1}^{n} p_i \\
I_2 &= \frac{1}{2}Y_1^2 - Y_2 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \\
I_3 &= \frac{1}{3}Y_1^3 - Y_1 Y_2 + Y_3 = \frac{1}{3}\sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n-1} (p_i + p_{i+1})\,\mathrm{e}^{q_i - q_{i+1}} \\
I_4 &= \frac{1}{4}Y_1^4 - Y_1^2 Y_2 + \frac{1}{2}Y_2^2 + Y_1 Y_3 - Y_4 \\
&= \frac{1}{4}\sum_{i=1}^{n} p_i^4 + \sum_{i=1}^{n-1} \left(p_i^2 + 2p_i p_{i+1} + p_{i+1}^2\right) \mathrm{e}^{q_i - q_{i+1}} + \frac{1}{2}\sum_{i=1}^{n-1} \mathrm{e}^{2(q_i - q_{i+1})} + \sum_{i=1}^{n-2} \mathrm{e}^{q_i - q_{i+2}} \\
I_m &= (-1)^m Y_m + m^{-1} \sum_{k=1}^{m-1} (-1)^k I_{m-k} Y_k.
\end{aligned} \tag{18}$$

To shed more light on expression (7) let us calculate Y_1 in detail. Calculating multivector fields W^n and $\hat{W} \wedge W^{n-1}$ gives rise to

$$W^n = n \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2} \wedge \dots \wedge \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_n} \tag{19}$$

and

$$\hat{W} \wedge W^{n-1} = \sum_{k=1}^{n} p_k \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2} \wedge \dots \wedge \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_n} \tag{20}$$

while the ratio of these bivector fields reproduces the total momentum of the Toda chain. The condition $[[E[E, W]]W] = 0$ satisfied by the generator of the symmetry E ensures that the conservation laws are in involution, i.e. $\{Y_k, Y_m\} = 0$. Thus the conservation laws as well as the bi-Hamiltonian structure of the non-periodic Toda chain appear to be associated with non-Noether symmetry. Some other non-Noether symmetries of the Toda chain were discussed in [5].

Unlike the Toda model the dynamical systems in our next examples are infinite dimensional and in order to ensure integrability one should construct an infinite number of conservation laws. Fortunately in several integrable models this task could be effectively done by identifying the appropriate non-Noether symmetry. First let us consider the well-known infinite-dimensional integrable Hamiltonian system—the nonlinear Schrödinger equation (NSE)

$$\psi_t = \mathrm{i}(\psi_{xx} + 2\psi^2\bar{\psi}) \tag{21}$$

where ψ is a smooth complex function of $(t, x) \in R^2$. At this stage we will not specify any boundary conditions and will just focus on symmetries of the NSE. Supposing that the vector field E generates the symmetry of the NSE one gets the following restriction,

$$E(\psi)_t = \mathrm{i}[E(\psi)_{xx} + 2\psi^2 E(\bar{\psi}) + 4\psi\bar{\psi}E(\psi)] \tag{22}$$

(obtained by substituting infinitesimal transformation $\psi \to \psi + aE(\psi) + O(a^2)$ generated by E into the NSE). It appears that the NSE possesses a nontrivial symmetry that is generated by the vector field

$$E(\psi) = \mathrm{i}\left(\psi_x + \frac{x}{2}\psi_{xx} + \psi\phi + x\psi^2\bar{\psi}\right) - t(\psi_{xxx} + 6\psi\bar{\psi}\psi_x) \tag{23}$$

(here ϕ is defined by $\phi_x = \psi\bar{\psi}$). In order to construct conservation laws we also need to know the Poisson bracket structure and it appears that the invariant Poisson bivector field could be defined if ψ is subjected to either periodic $\psi(t, -\infty) = \psi(t, +\infty)$ or zero $\psi(t, -\infty) = \psi(t, +\infty) = 0$ boundary conditions. In terms of variational derivatives the explicit form of the Poisson bivector field is

$$W = \mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\delta}{\delta\psi} \wedge \frac{\delta}{\delta\bar{\psi}} \tag{24}$$

while the corresponding symplectic form obtained by inverting W is

$$\omega = \mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x\, \delta\psi \wedge \delta\bar{\psi}. \tag{25}$$

Now one can check that the NSE could be rewritten in Hamiltonian form

$$\psi_t = \{h, \psi\} \tag{26}$$

with Poisson bracket $\{,\}$ defined by W and

$$h = \int_{-\infty}^{+\infty} \mathrm{d}x(\psi^2\bar{\psi}^2 - \psi_x\bar{\psi}_x). \tag{27}$$

Knowing the symmetry of the NSE that appears to be non-Noether ($[E, W] \neq 0$) one can construct bi-Hamiltonian structure and conservation laws. First let us calculate the Lie derivative of symplectic form along the symmetry generator

$$L_E\omega = \int_{-\infty}^{+\infty} [\delta\psi_x \wedge \delta\bar{\psi} + \psi\delta\phi \wedge \delta\bar{\psi} + \bar{\psi}\delta\phi \wedge \delta\psi]\,\mathrm{d}x. \tag{28}$$

The couple of 2-forms ω and $L_E\omega$ exactly reproduces the bi-Hamiltonian structure of the NSE proposed by Magri [4] while the conservation laws associated with this symmetry are well-known conservation laws of the NSE

$$I_1 = Y_1 = 2\int_{-\infty}^{+\infty} \psi\bar{\psi}\,\mathrm{d}x$$

$$I_2 = Y_1^2 - 2Y_2 = \mathrm{i}\int_{-\infty}^{+\infty} (\bar{\psi}_x\psi - \psi_x\bar{\psi})\,\mathrm{d}x$$

$$I_3 = Y_1^3 - 3Y_1Y_2 + 3Y_3 = 2\int_{-\infty}^{+\infty} (\psi^2\bar{\psi}^2 - \psi_x\bar{\psi}_x)\,\mathrm{d}x$$

$$
\begin{aligned}
I_4 &= Y_1^4 - 4Y_1^2Y_2 + 2Y_2^2 + 4Y_1Y_3 - 4Y_4 \\
&= \int_{-\infty}^{+\infty} [\mathrm{i}(\bar{\psi}_x\psi_{xx} - \psi_x\bar{\psi}_{xx}) + 3\mathrm{i}(\bar{\psi}\psi^2\bar{\psi}_x - \psi\bar{\psi}^2\psi_x)]\,\mathrm{d}x \\
I_m &= (-1)^m mY_m + \sum_{k=1}^{m-1}(-1)^k I_{m-k}Y_k.
\end{aligned}
\tag{29}
$$

The involutivity of the conservation laws of the NSE $\{Y_k, Y_m\} = 0$ is related to the fact that E satisfies the Yang–Baxter equation $[[E[E, W]]W] = 0$. Another class of symmetries of the NSE was discussed in [6].

Now let us consider other important integrable models—the Korteweg–de Vries equation (KdV) and modified Korteweg–de Vries equation (mKdV). Here symmetries are more complicated but the generator of the symmetry can still be identified and used in the construction of conservation laws. The KdV and mKdV equations have the following form:

$$u_t + u_{xxx} + uu_x = 0\,[\mathrm{KdV}] \tag{30}$$

and

$$u_t + u_{xxx} - 6u^2u_x = 0\,[\mathrm{mKdV}] \tag{31}$$

(here u is a smooth function of $(t, x) \in R^2$). The generators of symmetries of KdV and mKdV should satisfy conditions

$$E(u)_t + E(u)_{xxx} + u_xE(u) + uE(u)_x = 0\,[\mathrm{KdV}] \tag{32}$$

and

$$E(u)_t + E(u)_{xxx} - 12uu_xE(u) - 6u^2E(u)_x = 0\,[\mathrm{mKdV}] \tag{33}$$

(again these conditions are obtained by substituting infinitesimal transformation $u \to u + aE(u) + O(a^2)$ into KdV and mKdV, respectively). Further we will focus on the symmetries generated by the following vector fields,

$$
\begin{aligned}
E(u) = \frac{1}{2}u_{xx} + \frac{1}{6}u^2 + \frac{1}{24}u_xv + \frac{x}{8}(u_{xxx} + uu_x) \\
- \frac{t}{16}(6u_{xxxxx} + 20u_xu_{xx} + 10uu_{xxx} + 5u^2u_x)\,[\mathrm{KdV}]
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
E(u) = -\frac{3}{2}u_{xx} + 2u^3 + u_xw - \frac{x}{2}(u_{xxx} - 6u^2u_x) \\
- \frac{3t}{2}\left(u_{xxxxx} - 10u^2u_{xxx} - 40uu_xu_{xx} - 10u_x^3 + 30u^4u_x\right)[\mathrm{mKdV}]
\end{aligned}
\tag{35}
$$

(here v and w are defined by $v_x = u$ and $w_x = u^2$). To construct conservation laws we need to know the Poisson bracket structure and again as in the case of the NSE the Poisson bivector field is well defined when u is subjected to either periodic $u(t, -\infty) = u(t, +\infty)$ or zero $u(t, -\infty) = u(t, +\infty) = 0$ boundary conditions. For both KdV and mKdV the Poisson bivector field is

$$W = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\delta}{\delta u} \wedge \frac{\delta}{\delta v} \tag{36}$$

with the corresponding symplectic form

$$\omega = \int_{-\infty}^{+\infty} \mathrm{d}x\, \delta u \wedge \delta v \tag{37}$$

leading to the Hamiltonian realization of the KdV and mKdV equations

$$u_t = \{h, u\} \tag{38}$$

with Hamiltonians

$$h = \int_{-\infty}^{+\infty} \left(u_x^2 - \frac{u^3}{3} \right) \mathrm{d}x \text{ [KdV]} \tag{39}$$

and

$$h = \int_{-\infty}^{+\infty} \left(u_x^2 + u^4 \right) \mathrm{d}x \text{ [mKdV]}. \tag{40}$$

By taking the Lie derivative of the symplectic form along the generators of the symmetries one gets another couple of symplectic forms

$$L_E\omega = \int_{-\infty}^{+\infty} \mathrm{d}x \left(\delta u \wedge \delta u_x + \tfrac{2}{3} u \delta u \wedge \delta v \right) \text{ [KdV]} \tag{41}$$

$$L_E\omega = \int_{-\infty}^{+\infty} \mathrm{d}x \left(\delta u \wedge \delta u_x - 2u \delta u \wedge \delta w \right) \text{ [mKdV]} \tag{42}$$

involved in the bi-Hamiltonian realization of KdV/mKdV hierarchies and proposed by Magri [4]. The conservation laws associated with the symmetries reproduce an infinite sequence of conservation laws of the KdV equation

$$\begin{aligned}
I_1 &= Y_1 = \frac{2}{3} \int_{-\infty}^{+\infty} u \, \mathrm{d}x \\
I_2 &= Y_1 - 2Y_2 = \frac{4}{9} \int_{-\infty}^{+\infty} u^2 \, \mathrm{d}x \\
I_3 &= Y_1^3 - 3Y_1 Y_2 + 3Y_3 = \frac{8}{9} \int_{-\infty}^{+\infty} \left(\frac{u^3}{3} - u_x^2 \right) \mathrm{d}x \\
I_4 &= Y_1^4 - 4Y_1^2 Y_2 + 2Y_2^2 + 4Y_1 Y_3 - 4Y_4 \\
&= \frac{64}{45} \int_{-\infty}^{+\infty} \left(\frac{5}{36} u^4 - \frac{5}{3} u u_x^2 + u_{xx}^2 \right) \mathrm{d}x \\
I_m &= (-1)^m m Y_m + \sum_{k=1}^{m-1} (-1)^k I_{m-k} Y_k
\end{aligned} \tag{43}$$

and the mKdV equation

$$\begin{aligned}
I_1 &= Y_1 = -4 \int_{-\infty}^{+\infty} u^2 \, \mathrm{d}x \\
I_2 &= Y_1 - 2Y_2 = 16 \int_{-\infty}^{+\infty} \left(u^4 + u_x^2 \right) \mathrm{d}x \\
I_3 &= Y_1^3 - 3Y_1 Y_2 + 3Y_3 = -32 \int_{-\infty}^{+\infty} \left(2u^6 + 10u^2 u_x^2 + u_{xx}^2 \right) \mathrm{d}x \\
I_4 &= Y_1^4 - 4Y_1^2 Y_2 + 2Y_2^2 + 4Y_1 Y_3 - 4Y_4 \\
&= \frac{256}{5} \int_{-\infty}^{+\infty} \left(5u^8 + 70u^4 u_x^2 - 7u_x^4 + 14u^2 u_{xx}^2 + u_{xxx}^2 \right) \mathrm{d}x \\
I_m &= (-1)^m m Y_m + \sum_{k=1}^{m-1} (-1)^k I_{m-k} Y_k
\end{aligned} \tag{44}$$

The involutivity of these conservation laws is well known and in terms of the symmetry generators it is ensured by conditions $[[E[E, W]]W] = 0$. Thus the conservation laws and bi-Hamiltonian structures of KdV and mKdV hierarchies are related to the non-Noether symmetries of KdV and mKdV equations.

The purpose of the present paper was to illustrate some features of non-Noether symmetries discussed in [1] and to show that in several important integrable models the existence of complete sets of conservation laws could be related to such symmetries.

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